

of the pricing remains the factor copula approach, which is associated with semi-analytical pricing techniques and easy to use large portfolio approximations. This we believe is an alternative (or complementary) to the market standard base correlation approach.

It can be seen that distribution functions of conditional default probabilities are unambiguous modeling inputs, while local correlation suffers from existence and non-uniqueness issues. Thus, a local correlation approach offers some intuition but is ultimately not an obvious way to parameterize a model.

Stochastic and local correlation approaches suffer from global consistency, related to calibration to different maturities, credit spread dynamics and hedging issues. Such issues may well need to be addressed outside a copula framework. Before this, however, the copula skew approach is a valuable next step in understanding the pricing and risk management issues of bespoke credit portfolios. Stochastic and local correlation might thus either appear as transient and transitional since they do not sever the umbilical cord with the Gaussian copula and lead towards a mature modeling framework. But they do, we believe, give a useful way in which to analyse the problems of the correlation skew, move away from base correlation approaches and take the next step forward in correlation modeling.

APPENDIX A – Distribution of conditional default probabilities within the stochastic correlation model

$$Q(p_t^{V, B_s} \leq p) = (1 - q_s) E \left\{ Q \left[(1 - q) \Phi \left(\frac{K_t - \rho V}{\sqrt{1 - \rho^2}} \right) + q \Phi(K_t \leq p | V) \right] \right\} + q_s E \left[Q(1_{V \leq K_t} \leq p | V) \right]$$

for $0 \leq p < 1$. $1_{V \leq K_t} \leq p \Leftrightarrow V > K_t$. As a consequence, $E[Q(1_{V \leq K_t} \leq p | V)] = \Phi(-K_t)$. Let us assume that $q < 1, \rho > 0$. Then,

$$(1 - q) \Phi \left(\frac{K_t - \rho V}{\sqrt{1 - \rho^2}} \right) + q \Phi(K_t)$$

varies between $q \Phi(K_t)$ and $1 - q + q \Phi(K_t)$. We recall that $\Phi(K_t) = F(t)$.

□ if $qF(t) < p < 1 - q + qF(t)$,

$$(1 - q) \Phi \left(\frac{K_t - \rho V}{\sqrt{1 - \rho^2}} \right) + q \Phi(K_t) \leq p \Leftrightarrow -V \leq \frac{1}{\rho} \left[\sqrt{1 - \rho^2} \Phi^{-1} \left(\frac{p - q \Phi(K_t)}{1 - q} \right) - K_t \right]$$

Then

$$Q \left[(1-q) \Phi \left(\frac{K_t - \rho V}{\sqrt{1-\rho^2}} \right) + q \Phi(K_t) \leq p \mid V \right] = 1$$

$$\text{if } -V \leq \frac{1}{\rho} \left[\sqrt{1-\rho^2} \Phi^{-1} \left(\frac{p-q\Phi(K_t)}{1-q} \right) - K_t \right]$$

and 0 otherwise.

□ if $p \leq qF(t)$, then

$$Q \left[(1-q) \Phi \left(\frac{K_t - \rho V}{\sqrt{1-\rho^2}} \right) + q \Phi(K_t) \leq p \mid V \right] = 0$$

□ if $p \geq 1-q+qF(t)$, then

$$Q \left[(1-q) \Phi \left(\frac{K_t - \rho V}{\sqrt{1-\rho^2}} \right) + q \Phi(K_t) \leq p \mid V \right] = 1$$

As a consequence, we can write

$$E \left\{ Q \left[(1-q) \Phi \left(\frac{K_t - \rho V}{\sqrt{1-\rho^2}} \right) + q \Phi(K_t) \leq p \mid V \right] \right\}$$

as:

$$1_{qF(t) < p < 1-q+qF(t)} \Phi \left\{ \frac{1}{\rho} \left[\sqrt{1-\rho^2} \Phi^{-1} \left(\frac{p-q\Phi(K_t)}{1-q} \right) - K_t \right] \right\} + 1_{p \geq 1-q+qF(t)}$$

This leads to the stated expression.

APPENDIX B – Computation of zero-coupon CDO premiums

We detail here the computation of zero-coupon CDO premiums using the large homogeneous portfolio approximation. We need to compute

$$\int (p-k)^+ dG(p) = E \left[\left(p_t^{\cdot |V, B_s} - k \right)^+ \right]$$

$$= q_s E \left[\left(p_t^{\cdot |V, 1} - k \right)^+ \right] + (1-q_s) E \left[\left(p_t^{\cdot |V, 0} - k \right)^+ \right]$$

Since

$$p_t^{\cdot |V, 1} = 1_{V \leq \Phi^{-1}(F(t))}, \quad E \left[\left(p_t^{\cdot |V, 1} - k \right)^+ \right] = (1-k) F(t)$$

Let us now compute

$$E\left[\left(p_t^{|V,0} - k\right)^+\right]$$

where

$$p_t^{|V,0} = (1-q)\Phi\left(\frac{\Phi^{-1}(F(t)) - \rho V}{\sqrt{1-\rho^2}}\right) + qF(t)$$

This can be written as

$$(1-q)E\left\{\left[\Phi\left(\frac{\Phi^{-1}(F(t)) - \rho V}{\sqrt{1-\rho^2}}\right) - k^*\right]^+\right\}$$

with

$$k^* = \frac{k - qF(t)}{1-q}$$

We will further focus on the case where $0 < k^* < 1$ or equivalently where $qF(t) < k < 1 - q + qF(t)$, the other cases leading to straightforward computations.

$$\Phi\left(\frac{\Phi^{-1}(F(t)) - \rho V}{\sqrt{1-\rho^2}}\right) > k^*$$

if and only if

$$V < V^* = \frac{1}{\rho}\Phi\left[\Phi^{-1}(F(t)) - \sqrt{1-\rho^2}\Phi^{-1}(k^*)\right]$$

Let us remark that

$$\Phi\left(\frac{\Phi^{-1}(F(t)) - \rho V}{\sqrt{1-\rho^2}}\right)$$

can be written as $E[1_{W_i \leq \Phi^{-1}F(t)} | V]$ where

$$W_i = \rho V + \sqrt{1-\rho^2}\bar{V}_i$$

As a consequence,

$$E\left\{\left[\Phi\left(\frac{\Phi^{-1}(F(t)) - \rho V}{\sqrt{1-\rho^2}}\right) - k^*\right]^+\right\} = \Phi_{2,\rho}[\Phi^{-1}(F(t)), V^*] - k^*\Phi(V^*)$$

APPENDIX C – Computation of the marginal compound correlation

Let us consider the inverse problem of finding a marginal compound correlation from the distribution of conditional default probabilities G . To ease the discussion, we assume here that $\delta = 0$. As an example of the difficulties involved, let us assume that $F(t) = 0.4$ and G being associated with a discrete distribution with probability mass of 0.5 at 0 and 0.5 at 0.8.²⁷ For $0.5 < \alpha < 0.8$, $G(\alpha) = 0.5$, thus $G(\alpha) < 1 - F(t)$ and there is then no solution to equation (10).²⁸ Unfortunately, marginal compound correlation may not exist for consistent pricing models.

Equation (10) can equivalently be written as:

$$\bar{\rho}(\alpha)\Phi^{-1}(G(\alpha)) + \Phi^{-1}(F(t)) = \sqrt{1 - \bar{\rho}(\alpha)^2}\Phi^{-1}(\alpha)$$

We recall that for notational simplicity, $\delta = 0$. By squaring up the previous equality, we obtain the following second order equation, from which we must consider the positive roots:

$$\begin{aligned} (\Phi^{-1}(G(\alpha))^2 + \Phi^{-1}(\alpha)^2)\bar{\rho}^2 + 2\Phi^{-1}(G(\alpha))\Phi^{-1}(F(t))\bar{\rho} + \\ \Phi^{-1}(F(t))^2 - \Phi^{-1}(\alpha)^2 = 0 \quad (C1) \end{aligned}$$

Let us emphasize that any solution of:

$$\bar{\rho}(\alpha)\Phi^{-1}(G(\alpha)) + \Phi^{-1}(F(t)) = -\sqrt{1 - \bar{\rho}(\alpha)^2}\Phi^{-1}(\alpha) \quad (C2)$$

is also a solution of equation (C1). The roots of equation (C1) are the roots of equation (10) plus the roots of equation (C2). In the numerical examples below, we firstly write the roots of equation (C1) and then check that they indeed solve equation (10).

Let us discuss the existence of a solution to equation (C1). We denote by:

$$\Delta' = \Phi^{-1}(\alpha)^2 \times [\Phi^{-1}(G(\alpha))^2 + \Phi^{-1}(\alpha)^2 - \Phi^{-1}(F(t))^2]$$

Equation (C1) has real roots if and only if $\Delta' \geq 0$ but this may not be the case as illustrated by the following example. We assume that $p_i^{i|V}$ is discretely distributed with probability 0.5 to be equal to 0 and 0.5 to be equal to 0.5. This leads to a marginal default probability of 0.25. G is constant on $]0, 0.5[$ and equal to 0.5. Then $\Delta' < 0$ for $\alpha \in]0.25, 0.5[$.

Let us now discuss whether equation (C1) admits a unique solution. If $\Phi^{-1}(F(t))^2 > \Phi^{-1}(\alpha)^2$, and provided that $\Delta' > 0$, there are two real roots of the same sign. We previously showed some practical cases where they were actually two positive roots to equation (C1) that appear to be also solutions of equation (10).

²⁷ It can be checked that $\int_0^1 p dG(p) = F(t)$ which means that the expected conditional default probability is the marginal default probability.

²⁸ The right-hand term of equation (10) decreases from 1 to $1 - F(t)$ as $\bar{\rho}(\alpha)$ increases from 0 to 1.

APPENDIX D – Derivation of the local correlation function

We denote by:

$$\varepsilon(V) = \frac{H^{-1}(F(t)) + \rho(V)V}{\sqrt{1 - \rho^2(V)}}$$

and assume that ε is increasing with V . From equation (17), we have $p_t^i | V = \Phi(\varepsilon(V))$. Thus, for p in $[0, 1]$, $G(p) = Q(p_t^i | V \leq p) = \Phi(\varepsilon^{-1}(\Phi^{-1}(p)))$, which leads to $\varepsilon(v) = \Phi^{-1}(G^{-1}(\Phi(v)))$ for $v \in \mathbb{R}$. We thus need to look for a function ρ such that:

$$\frac{H^{-1}(F(t)) + \rho(v)v}{\sqrt{1 - \rho^2(v)}} = \Phi^{-1}[G^{-1}(\Phi(v))], \forall v \in \mathbb{R} \quad (D1)$$

with:

$$H(x) = \int_{\mathbb{R}} \Phi\left(\frac{x + \rho(v)v}{\sqrt{1 - \rho^2(v)}}\right) \varphi(v) dv$$

The right-hand term is given while the left-hand term involves ρ either directly or indirectly through H .

We proceed with a fixed point algorithm, constructing a series of functions ρ_n converging to a solution of previous functional equation. We set $\rho_0(v) = \rho$, where ρ is some arbitrary correlation parameter (ρ_0 is then a constant function). We then set H_0 such that

$$H_0(x) = \int_{\mathbb{R}} \Phi\left(\frac{x + \rho v}{\sqrt{1 - \rho^2}}\right) \varphi(v) dv$$

which leads obviously to $H_0 = \Phi$. We then solve the following equation for all v s:

$$\frac{\Phi^{-1}(F(t)) + \rho_1(v)v}{\sqrt{1 - \rho_1^2(v)}} = \Phi^{-1}[G^{-1}(\Phi(v))] \quad (D2)$$

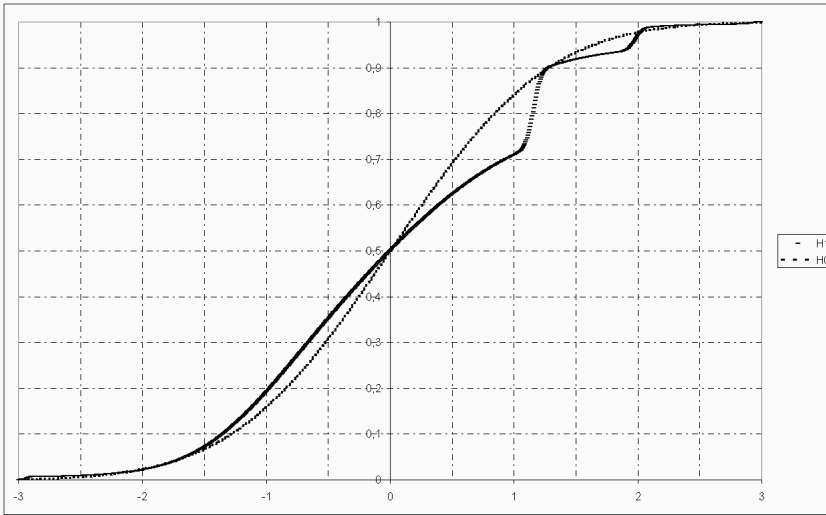
which provides ρ_1 .²⁹ H_1 is such that

$$H_1(x) = \int_{\mathbb{R}} \Phi\left(\frac{x + \rho_1(v)v}{\sqrt{1 - \rho_1^2(v)}}\right) \varphi(v) dv$$

Step n simply involves solving for $\rho_n(v)$ in:

²⁹ The previous equation is quite similar to the one that provides the marginal compound correlation. By squaring up, we obtain a second order algebraic equation, which we solve as usual.

FIGURE 9 Check of convergence of the fixed-point algorithm (stochastic correlation model).



$$\frac{H_{n-1}^{-1}(F(t)) + \rho_n(v)v}{\sqrt{1 - \rho_n^2(v)}} = \Phi^{-1}[G^{-1}(\Phi(v))]$$

and then set H_n such that

$$H_n(x) = \int_{\mathbb{R}} \Phi\left(\frac{x + \rho_n(v)v}{\sqrt{1 - \rho_n^2(v)}}\right) \varphi(v) dv$$

Whenever $H_n = H_{n-1}$,³⁰ the algorithm has converged and we set $\rho = \rho_n$.

Equation (10) can be equivalently written as

$$\frac{\Phi^{-1}(F(t)) + \bar{\rho}(p)\Phi^{-1}(G(p))}{\sqrt{1 - \bar{\rho}(p)^2}} = \Phi^{-1}(p)$$

By stating $p = G^{-1}(\Phi(v))$, we obtain

$$\frac{\Phi^{-1}(F(t)) + \bar{\rho}(p)v}{\sqrt{1 - \bar{\rho}(p)^2}} = \Phi^{-1}[G^{-1}(\Phi(v))]$$

Comparing with equation (D2), we get $\bar{\rho}(p) = \rho_1(v)$, which means that local

³⁰ We only need to check that $H_n^{-1}(F(t)) = H_{n-1}^{-1}(F(t))$.